

# Multifractal analysis for historic set in topological dynamical systems

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**Abstract.** In this article, the historic set is divided into different level sets and we use topological pressure to describe the size of these level sets. We give an application of these results to dimension theory. Especially, we use topological pressure to describe the relative multifractal spectrum of ergodic averages and give a positive answer to the conjecture posed by L. Olsen (J. Math. Pures Appl. **82** (2003)).

**Keywords:** Topological pressure; historic set; multifractal analysis.

## 1 Introduction

A topological dynamical system is a triple  $(X, d, T)$  (or tuple  $(X, T)$  for short) consisting of a compact metric space  $(X, d)$  and a continuous map  $T : X \rightarrow X$ .

An orbit  $\{x, T(x), T^2(x), \dots\}$  has historic behavior if for some continuous function  $\psi : X \rightarrow \mathbb{R}$ , the average

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(T^i(x))$$

does not exist. This terminology was introduced by Ruelle in [26]. If this limit does not exist, it follows that 'partial averages'  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(T^i x)$  keep change considerably so that their values give information about the epoch to which  $n$  belongs.

The problem, whether there are persistent classes of smooth dynamical systems such that the set of initial states which give rise to orbits with historic behavior has 'positive Lebesgue measure' was discussed by Ruelle [26]. Takens also investigated the problem in the survey [27].

Very recently, the idea of multifractal analysis plays an important role in the study of dynamical system. V. Climenhaga [7] considered the topological pressure function on the level sets of asymptotically defined quantities in a topological dynamical systems. D. Feng and W. Huang [11] studied the general asymptotically sub-additive on general topological dynamical systems and established some variational relations between the topological entropy of the level sets of Lyapunov exponents, measure-theoretic entropies and topological pressures in this general situation.

In this article, we will use the framework introduced by Olsen to investigate the geometric structure of the historic set in view of multifractal analysis.

Denote by  $M(X)$ ,  $M(X, T)$  and  $E(X, T)$ , the set of all Borel probability measures on  $X$ , the collection of all  $T$ -invariant Borel probability measures, and the set of all ergodic  $T$ -invariant Borel probability measures, respectively.

It is well-known that  $M(X)$  and  $M(X, T)$  are both convex, compact spaces endowed with weak\* topology. For  $\mu, \nu \in M(X)$ , define a compatible metric  $\rho$  on  $M(X)$  as follows:

$$\rho(\mu, \nu) := \sum_{k \geq 1} \frac{|\int_X f_k d\mu - \int_X f_k d\nu|}{2^k},$$

where  $\{f_1, f_2, \dots\}$  is a countable and dense in  $C(X, [0, 1])$ . Note that  $\rho(\mu, \nu) \leq 1$ , for any  $\mu, \nu \in M(X)$ . This article uses an equivalent metric on  $X$ , still denoted by  $d$ ,

$$d(x, y) := \rho(\delta_x, \delta_y)$$

for convenience. For  $n \in \mathbb{N}$ , let  $L_n : X \rightarrow M(X)$  be the  $n$ -th empirical measure, i.e.,

$$L_n x = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k x},$$

where  $\delta_x$  denotes the Dirac measure at  $x$ . Let  $\Xi$  be a continuous affine map from  $M(X)$  to a vector space  $Y$  with a linear compatible metric  $d'$ .  $(Y, \Xi)$  is called a deformation of  $L_n$ . Let  $A(x_n)$  be the set of accumulation points of  $\{x_n\}$  and  $D(T, \Xi)$  be the set consists of the points  $x$  such that  $\lim_n \Xi L_n x$  does not exist.  $D(T, \Xi)$  is called the historic set for  $(X, T)$ .

This article is devoted to investigate the structure of  $D(T, \Xi)$  via the following framework introduced and developed by Olsen [14], [15], [16], [17] and Olsen & Winter [18].

More precisely, for a subset  $C$  of  $Y$ , this article uses topological pressure to describe the size of the following so-called sup set, equ set and sub set:

$$\begin{aligned}\Delta_{sup}(C) &= \{x \in X : A(\Xi L_n x) \subset C\}, \\ \Delta_{equ}(C) &= \{x \in X : A(\Xi L_n x) = C\}, \\ \Delta_{sub}(C) &= \{x \in X : A(\Xi L_n x) \supset C\}.\end{aligned}$$

Such sets together give us a complete description of the dynamics of the historic set and provides the basis for a substantially better understanding of the underlying geometry of the historic set. More generally, for  $S_1, S_2 \subset Y$ , considering  $\Delta(S_1, S_2) = \{x \in X : S_1 \subset A(\Xi L_n x) \subset S_2\}$ , we have

$$\begin{aligned}\Delta(\emptyset, C) &= \Delta_{sup}(C); \\ \Delta(C, C) &= \Delta_{equ}(C); \\ \Delta(C, Y) &= \Delta_{sub}(C).\end{aligned}$$

Obviously, multifractal analysis is a special case of this framework. For example, for any  $\phi \in C(X, \mathbb{R})$ , choose  $Y = \mathbb{R}$ , and define  $\Xi : M(X) \rightarrow \mathbb{R}$  by  $\Xi : \mu \mapsto \int \phi d\mu$ . Then for  $C = \{\alpha\} \subseteq \mathbb{R}$ , it follows that

$$\Delta_{equ}(C) = \left\{ x \in X : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \alpha \right\}, \quad (1.1)$$

and

$$D(T, \Xi) = \left\{ x \in X : \text{the limit } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \text{ does not exist} \right\}. \quad (1.2)$$

Previous studies [2], [3], [9], [21], [25], [28], [29], [33] have obtained a number of fruitful results regarding different quantities to describe the size of (1.1) in some dynamical systems with some mixing properties. The quantities include Hausdorff dimension, packing entropy, topological entropy and topological pressure. Dynamical systems can be symbolic spaces or satisfy some mild conditions such as the specification property or the g-almost product property and so on.

At the beginning,  $D(T, \Xi)$  had been considered of little interest in dynamical systems and geometric measure theory due to the fact that  $\mu(D(T, \Xi)) = 0$  for any  $\mu \in M(X, T)$ . However, recent work [4], [5], [22], [30], [34] has changed such attitudes. Hausdorff dimension or topological entropy or topological pressure of (1.2) can be large enough even equal to that of the whole space. It illustrates that the historic set has rich information. Hence,

it is meaningful to divide the historic set into different level sets and investigate these level sets. A series results in symbolic space and iterated function system can be found in [1], [14], [15], [16], [17]. This article divide  $D(T, \Xi)$  into different level sets  $\Delta_{equ}(\cdot)$  and  $\Delta_{sub}(\cdot)$ .

This investigation uses topological pressure to describe  $\Delta_{equ}(\cdot)$ ,  $\Delta_{sub}(\cdot)$  and so on. Topological pressure is a powerful tool and is not only a generalization of topological entropy but also closely related to Hausdorff dimension. This article discusses the dynamical systems satisfying g-almost product property and the uniform separation property that were introduced by C. Pfister and W. Sullivan [25]. These two properties are strictly weaker than the specification property and the positive expansive property. (For example, all  $\beta$ -shifts have the g-almost product property and the uniform separation property is true for expansive and more generally asymptotically h-expansive maps.)

As an application of our results, we study symbolic spaces and iterated function systems. We stress that the metric in symbolic space here is ultrametric rather than the metric in [15].

For  $\varphi \in C(X, \mathbb{R})$ , define

$$\Lambda(y, \varphi) = \begin{cases} \sup_{\substack{\mu \in M(X, T) \\ \Xi \mu = y}} \{h(T, \mu) + \int \varphi d\mu\}, & \text{for } y \in \Xi(M(X, T)) \\ -\infty, & \text{otherwise.} \end{cases} \quad (1.3)$$

We state our main theorems as below:

**Theorem 1.1.**  *$(X, T, \Xi, L_n, Y)$  satisfies g-almost product property and the uniform separation property and  $\varphi \in C(X, \mathbb{R})$ . If*

1.  *$C \subset Y$  is not a compact and connected subset of  $\Xi(M(X, T))$ , then*

$$\{x \in X : A(\Xi L_n x) = C\} = \emptyset,$$

2.  *$C \subset Y$  is a compact and connected subset of  $\Xi(M(X, T))$ , then*

$$P(\Delta_{equ}(C), \varphi) = \inf_{y \in C} \sup_{\substack{\mu \in M(X, T) \\ \Xi \mu = y}} \left\{ h(T, \mu) + \int \varphi d\mu \right\} = \inf_{y \in C} \Lambda(y, \varphi).$$

**Theorem 1.2.**  *$(X, T, \Xi, L_n, Y)$  as before and  $\varphi \in C(X, \mathbb{R})$ . If*

1.  *$C \subset Y$  is not a subset of  $\Xi(M(X, T))$ , then*

$$\{x \in X : C \subset A(\Xi L_n x)\} = \emptyset,$$

2.  $C \subset Y$  is a subset of  $\Xi(M(X, T))$ , then

$$P(\Delta_{\text{sub}}(C), \varphi) = \inf_{y \in C} \sup_{\substack{\mu \in M(X, T) \\ \Xi\mu = y}} \left\{ h(T, \mu) + \int \varphi d\mu \right\} = \inf_{y \in C} \Lambda(y, \varphi).$$

**Theorem 1.3.**  $(X, T, \Xi, L_n, Y)$  as before and  $\varphi \in C(X, \mathbb{R})$ , fix  $S_1 \subset \Xi(M(X, T))$ ,  $S_2 \subset Y$ , if

1.  $S_1 = \emptyset$ , then

$$P(\Delta(S_1, S_2), \varphi) = \sup_{x \in S_2} \Lambda(x, \varphi),$$

2.  $S_1 \neq \emptyset$  and  $S_1$  is contained in a connected component of  $S_2$ , then

$$\sup_{\substack{S_1 \subset Q \subset S_2 \\ Q \subseteq \Xi(M(X, T)) \text{ is compact and connected}}} \inf_{x \in Q} \Lambda(x, \varphi) \leq P(\Delta(S_1, S_2), \varphi) \leq \inf_{x \in S_1} \Lambda(x, \varphi),$$

3.  $S_1 \neq \emptyset$  and  $S_1$  is not contained in a connected component of  $S_2$ , then

$$\{x \in X : S_1 \subset A(\Xi L_n x) \subset S_2\} = \emptyset.$$

**Theorem 1.4.**  $(X, T, \Xi, L_n, Y)$  as before and  $\varphi \in C(X, \mathbb{R})$ , fix  $S_1 \subset \Xi(M(X, T))$ ,  $S_2 \subset Y$ ,

1. If  $S_1 = \emptyset$ , then

$$P(\Delta(S_1, S_2), \varphi) = \sup_{x \in S_2} \Lambda(x, \varphi),$$

2. If  $S_1 \neq \emptyset$  and  $\overline{\text{co}}(S_1)$  the closed convex hull of  $S_1$  is contained in a connected component of  $S_2$ , then

$$P(\Delta(S_1, S_2), \varphi) = \inf_{x \in S_1} \Lambda(x, \varphi),$$

3. If  $S_1 \neq \emptyset$  and  $S_1$  is not contained in a connected component of  $S_2$ , then

$$\{x \in X : S_1 \subset A(\Xi L_n x) \subset S_2\} = \emptyset.$$

## 2 Preliminaries

A remark about notations is presented here for convenience.

**Remark 2.1.** Let  $(X, T)$  be a topological dynamical system.

- Let  $F \subset M(X)$  be a neighborhood, set  $X_{n,F} := \{x \in X : L_n x \in F\}$ .
- Given  $\delta > 0$  and  $\epsilon > 0$ , two points  $x$  and  $y$  are  $(\delta, n, \epsilon)$ -separated if  $\#\{j : d(T^j x, T^j y) > \epsilon, 0 \leq j \leq n-1\} \geq \delta n$ . A subset  $E$  is  $(\delta, n, \epsilon)$ -separated if any pair of different points of  $E$  are  $(\delta, n, \epsilon)$ -separated.
- Let  $F \subset M(X)$  be a neighborhood of  $\nu$ , and  $\epsilon > 0$ , set  
 $N(F; n, \epsilon) :=$  maximal cardinality of an  $(n, \epsilon)$ -separated subset of  $X_{n,F}$ ;  
 $N(F; \delta, n, \epsilon) :=$  maximal cardinality of an  $(\delta, n, \epsilon)$ -separated subset of  $X_{n,F}$ .
- Given  $x \in X$ , set  $B_n(x, \epsilon) := \{y \in X : d_n(x, y) \leq \epsilon\}$ , where  $d_n(x, y) = \max_{i=0, \dots, n-1} d(T^i x, T^i y)$ .
- A point  $x \in X$ ,  $\epsilon$ -shadows a sequence  $\{x_0, x_1, \dots, x_k\}$  if  $d(T^j x, x_j) \leq \epsilon \ \forall j = 0, 1, \dots, k$ .
- Let  $g : \mathbb{N} \rightarrow \mathbb{N}$  be a given nondecreasing unbound map with the properties  $g(n) < n$  and  $\lim_{n \rightarrow \infty} \frac{g(n)}{n} = 0$ . The function  $g$  is called blow-up function. Given  $x \in X$  and  $\epsilon > 0$ . The  $g$ -blow-up of  $B_n(x, \epsilon)$  is the closed set  
 $B_n(g; x, \epsilon) := \{y \in X : \exists \Lambda \subset \Lambda_n, \#(\Lambda_n \setminus \Lambda) \leq g(n) \text{ and } \max\{d(T^j x, T^j y) : j \in \Lambda\} \leq \epsilon\}$ ,  
where  $\Lambda_n = \{0, 1, \dots, n-1\}$ .
- (i) Given  $K \subset M(X, T)$ , set  $G_K := \{x \in X : A(L_n(x)) = K\}$ .  
(ii) Given  $K' \subset \Xi M(X, T)$ , set  $G_{K'}^* := \{x \in X : A(\Xi L_n(x)) = K'\}$ .  
(iii) Given  $K \subset M(X)$ , set  ${}^K G := \{x \in X : A(L_n(x)) \cap K \neq \emptyset\}$ .  
(iv) Given  $K' \subset Y$ , set  ${}^{K'} G^* := \{x \in X : A(\Xi L_n(x)) \cap K' \neq \emptyset\}$ .

**Definition 2.1.** [25] *The dynamical system  $(X, d, T)$  has the  $g$ -almost product property with blow-up function  $g$ , if there exists a nonincreasing function  $m : \mathbb{R}^+ \rightarrow \mathbb{N}$ , such that for any  $k \in \mathbb{N}$ , any  $x_1 \in X, \dots, x_k \in X$ , any positive  $\epsilon_1, \epsilon_2, \dots, \epsilon_k$  and any integers  $n_1 \geq m(\epsilon_1), \dots, n_k \geq m(\epsilon_k)$ ,*

$$\bigcap_{j=1}^k T^{-M_{j-1}} B_{n_j}(g; x_j, \epsilon_j) \neq \emptyset,$$

where  $M_0 = 0, M_i = n_1 + n_2 + \dots + n_i, i = 1, 2, \dots, k-1$ .

**Definition 2.2.** [25] *The dynamical system  $(X, d, T)$  has uniform separation property if for any  $\eta$ , there exist  $\delta^* > 0$  and  $\epsilon^* > 0$  so that for  $\mu$  ergodic and any neighborhood  $F \subset M(X)$  of  $\mu$ , there exists  $n_{F, \mu, \eta}^*$ , such that for  $n \geq n_{F, \mu, \eta}^*$ ,  $N(F; \delta^*, n, \epsilon^*) \geq \exp(n(h(T, \mu) - \eta))$ , where  $h(T, \mu)$  is the metric entropy of  $\mu$ .*

**Proposition 2.2.** [25] *Suppose that  $(X, d, T)$  has the  $g$ -almost product property. Let  $x_1, \dots, x_k \in X$ ,  $\epsilon_1 > 0, \dots, \epsilon_k > 0$ , and  $n_1 \geq m(\epsilon_1), \dots, n_k \geq m(\epsilon_k)$  be given. Assume that  $L_{n_j}(x_j) \in B(\nu_j, \zeta_j)$ ,  $0 \leq j \leq k$ . Then for any  $y \in \bigcap_{i=1}^k T^{-M_i-1} B_{n_i}(g; x_i, \epsilon_i)$  and any probability measure  $\alpha$ ,*

$$\rho(L_{M_k}(y), \alpha) \leq \sum_{j=1}^k \frac{n_j}{M_k} (\zeta'_j + \rho(\nu_j, \alpha)),$$

where  $M_j = n_1 + \dots + n_j$ ,  $\zeta'_j = \zeta_j + \epsilon_j + \frac{g(n_j)}{n_j}$ ,  $j = 1, \dots, k$ .

**Definition 2.3.** [23] *Given  $Z \subset X$ ,  $\varphi \in C(X, \mathbb{R})$ , and let  $\Gamma_n(Z, \epsilon)$  be the collection of all finite or countable covers of  $Z$  by sets of the form  $B_m(x, \epsilon)$ , with  $m \geq n$ . Let  $S_n \varphi(x) := \sum_{i=0}^{n-1} \varphi(T^i x)$ . Set*

$$M(Z, t, \varphi, n, \epsilon) := \inf_{\mathcal{C} \in \Gamma_n(Z, \epsilon)} \left\{ \sum_{B_m(x, \epsilon) \in \mathcal{C}} \exp \left( -tm + \sup_{y \in B_m(x, \epsilon)} S_m \varphi(y) \right) \right\},$$

and

$$M(Z, t, \varphi, \epsilon) = \lim_{n \rightarrow \infty} M(Z, t, \varphi, n, \epsilon),$$

then there exists a unique number  $P(Z, \varphi, \epsilon)$  such that

$$P(Z, \varphi, \epsilon) = \inf\{t : M(Z, t, \varphi, \epsilon) = 0\} = \sup\{t : M(Z, t, \varphi, \epsilon) = \infty\}.$$

$P(Z, \varphi) = \lim_{\epsilon \rightarrow 0} P(Z, \varphi, \epsilon)$  is called the topological pressure of  $Z$ .

It is obvious that the following hold:

1.  $P(Z_1, \varphi) \leq P(Z_2, \varphi)$  for any  $Z_1 \subset Z_2 \subset X$ ;
2.  $P(Z, \varphi) = \sup_i P(Z_i, \varphi)$ , where  $Z = \bigcup_i Z_i \subset X$ .

**Definition 2.4.** [5] *If  $Y$  is a vector space and  $d'$  is a metric in  $Y$ , then  $d'$  is linearly compatible if*

- (1) For all  $x_1, x_2, y_1, y_2 \in Y$ ,  $d'(x_1 + x_2, y_1 + y_2) \leq d'(x_1, y_1) + d'(x_2, y_2)$ .
- (2) For all  $x, y \in Y$  and all  $\lambda \in \mathbb{R}$ ,  $d'(\lambda x, \lambda y) \leq |\lambda| d'(x, y)$ .

In fact, if  $d'$  is induced by a norm, then  $d'$  is linearly compatible.

Now, we present several propositions about metrics  $d'$  and  $\rho$ .

**Proposition 2.3.** *Assume that  $d'$  is a linearly compatible metric in  $Y$ . Let*

$$V(\Xi, \epsilon) := \sup_{\substack{\mu, \nu \in M(X) \\ \rho(\mu, \nu) < \epsilon}} d'(\Xi\mu, \Xi\nu),$$

*then  $V(\Xi, \epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$ .*

*Proof.* It is because that  $\Xi : M(X) \rightarrow Y$  is continuous and  $M(X)$  is compact.  $\square$

**Proposition 2.4.** *For any  $x \in X$ , any  $\epsilon > 0$ , there exist sufficiently large  $N$ , such that for all  $n > N$ , we have  $\rho(L_n x, L_{n+1} x) \leq \epsilon$ .*

*Proof.* Choose sufficiently large  $N$ , such that  $\frac{1}{N+1} \leq \epsilon$ . Then

$$\begin{aligned} \rho(L_n x, L_{n+1} x) &= \sum_{k \geq 1} 2^{-k} \left| \int f_k dL_n x - \int f_k dL_{n+1} x \right| \\ &= \sum_{k \geq 1} 2^{-k} \left| \int f_k dL_n x - \int \frac{n}{n+1} f_k dL_n x - \int \frac{1}{n+1} f_k d\delta_{T^n x} \right| \\ &= \sum_{k \geq 1} 2^{-k} \frac{1}{n+1} \left| \int f_k dL_n x - \int f_k d\delta_{T^n x} \right| \\ &= \frac{1}{n+1} \rho(L_n x, \delta_{T^n x}) \\ &\leq \frac{1}{n+1} \leq \frac{1}{N+1} \leq \epsilon. \end{aligned}$$

$\square$

**Proposition 2.5.** *For any  $x \in X$ , any  $\epsilon > 0$ , there exist sufficiently large  $N$ , such that for all  $n > N$ , we have  $x' \in B_n(g, x, \epsilon)$  implies  $\rho(L_n x, L_n x') < 2\epsilon$ .*

*Proof.* Since  $\lim_{n \rightarrow \infty} \frac{g(n)}{n} = 0$ , we have that for any  $\epsilon > 0$ , there exists a sufficiently large  $N \in \mathbb{N}$  such that  $\frac{g(n)}{n} < \epsilon$  whenever  $n > N$ . Then

$$\begin{aligned} \rho(L_n x, L_n x') &= \sum_{k \geq 1} 2^{-k} \left| \int f_k dL_n x - \int f_k dL_n x' \right| \\ &\leq \sum_{k \geq 1} 2^{-k} \frac{g(n)}{n} + \frac{n - g(n)}{n} \epsilon \\ &\leq 2\epsilon. \end{aligned}$$

$\square$

**Proposition 2.6.** [32] *For any  $x \in X$ ,  $A(\Xi L_n x)$  is a compact and connected subset of  $Y$ .*



### 3 Upper and lower bounds for $P(G_{K'}^*, \varphi)$

This section is to show the upper and lower bounds for  $P(G_{K'}^*, \varphi)$ .

**Proposition 3.1.** [21] *Let  $(X, d, T)$  be a dynamical system,*

(i) *if  $K \subseteq M(X, T)$  is a closed subset, then*

$$P(KG, \varphi) \leq \sup \left\{ h(T, \mu) + \int \varphi d\mu : \mu \in K \right\}.$$

(ii) *if  $\mu \in M(X, T)$ , then*

$$P(G_\mu, \varphi) \leq h(T, \mu) + \int \varphi d\mu.$$

(iii) *if  $K \subseteq M(X, T)$  is a non-empty closed set, then*

$$P(G_K, \varphi) \leq \inf \left\{ h(T, \mu) + \int \varphi d\mu : \mu \in K \right\}.$$

By the above proposition, we get the following theorem.

**Theorem 3.1.** *Let  $(X, d, T)$  be a dynamical system,*

(1) *if  $K' \subset Y$  is a closed subset, then*

$$P(K'G^*, \varphi) \leq \sup \{ \Lambda(y, \varphi) : y \in K' \},$$

(2) *if  $K' \subset Y$  is a non-empty closed set, then*

$$P(G_{K'}^*, \varphi) \leq \inf \{ \Lambda(y, \varphi) : y \in K' \}.$$

*Proof.* (1) If  $K' \subset Y$  is a closed subset, then

$$\begin{aligned} K'G^* &= \left\{ x \in X : A(\Xi(L_n x)) \cap K' \neq \emptyset \right\} \\ &= \left\{ x \in X : \Xi(AL_n x) \cap K' \neq \emptyset \right\} \\ &= \left\{ x \in X : (AL_n x) \cap \Xi^{-1}K' \neq \emptyset \right\} \\ &= \Xi^{-1}K' \cap M(X, T) \quad G. \end{aligned}$$

hence,

$$\begin{aligned} P(K'G^*, \varphi) &= P(\Xi^{-1}K' \cap M(X, T)G, \varphi) \\ &\leq \sup \left\{ h(T, \mu) + \int \varphi d\mu : \mu \in \Xi^{-1}K' \cap M(X, T) \right\} \\ &= \sup \{ \Lambda(y, \varphi) : y \in K' \}. \end{aligned}$$

(2) If  $K' \not\subseteq \Xi M(X, T)$ , then  $G_{K'}^* = \emptyset$ . In this case, there exists  $y \in K' \setminus \Xi M(X, T)$  such that  $\Lambda(y, \varphi) = -\infty$ . If  $K' \subseteq \Xi M(X, T)$ , then for any  $y \in K'$ , we have

$$\begin{aligned} G_{K'}^* &= \{x \in X : A(\Xi L_n x) = K'\} \\ &\subseteq \left\{x \in X : A(\Xi L_n x) \cap \{y\} \neq \emptyset\right\} \\ &=_{\{y\}} G^*. \end{aligned}$$

Then,

$$P(G_{K'}^*, \varphi) \leq P(\{y\}G^*, \varphi) \leq \Lambda(y, \varphi),$$

for any  $y \in K'$ . In summary,

$$P(G_{K'}^*, \varphi) \leq \inf\{\Lambda(y, \varphi) : y \in K'\}.$$

□

To obtain the lower bound of  $P(G_{K'}^*, \varphi)$ , we need endow dynamical system with some mild conditions.

**Proposition 3.2.** [25] *Assume that  $(X, d, T)$  has the  $g$ -almost product property and the uniform separation property. For any  $\eta$ , there exists  $\delta^*$  and  $\epsilon^* > 0$  such that for  $\mu \in M(X, T)$  and any neighborhood  $F \subset M(X)$  of  $\mu$ , there exists  $n_{F, \mu, \eta}^*$ , such that*

$$N(F; \delta^*, n, \epsilon^*) \geq \exp(n(h(T, \mu) - \eta)),$$

whence  $n \geq n_{F, \mu, \eta}^*$ . Furthermore, for any  $\mu \in M(X, T)$ ,

$$h(T, \mu) \leq \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \inf_{F \ni \mu} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N(F; \delta, n, \epsilon).$$

**Lemma 3.1.** [25] *If  $K'$  is a connected, non-empty and compact subset of  $\Xi(M(X, T))$ , then there exists a sequence  $\{\alpha_1'', \alpha_2'', \dots\}$  in  $K'$  such that*

$$\overline{\{\alpha_j'' : j \in \mathbb{N}, j > n\}} = K',$$

for any  $n \in \mathbb{N}$ , and  $\lim_{j \rightarrow \infty} d'(\alpha_j'', \alpha_{j+1}'') = 0$ .

**Theorem 3.2.** *Let  $(X, d, T)$  be a dynamical system with the uniform separation and  $g$ -almost product property,  $\varphi \in C(X, \mathbb{R})$ . If  $K'$  is a connected, non-empty and compact subset of  $\Xi(M(X, T))$ , then*

$$\inf_{y \in K'} \Lambda(y, \varphi) \leq P(G_{K'}^*, \varphi).$$

*Proof.* Let  $\eta > 0$  and  $h^* := \inf_{y \in K'} \Lambda(y, \varphi) - \eta$ . For any  $s < h^*$ , set  $h^* - s := 2\delta > 0$ . Given sequence  $\{\alpha_k''\}$  as in Lemma 3.1, we construct a subset  $G$  such that for each  $x \in G$ ,  $\{\Xi L_n x\}$  has the same limit-point set as the sequence  $\{\alpha_k''\}$ , and  $P(G, \varphi) \geq h^*$ . For  $\frac{\eta}{2}$  and  $\alpha_k'' \in K'$ , there exists  $\alpha_k \in \Xi^{-1}\{\alpha_k''\} \subseteq M(X, T)$  such that  $\Lambda(\alpha_k'', \varphi) \leq h(T, \alpha_k) + \int \varphi d\alpha_k + \frac{\eta}{2}$ . By Proposition 3.2, it is easy to obtain that for  $\frac{\eta}{2} > 0$ , there exist  $\delta^* > 0$  and  $\epsilon^* > 0$  such that for any neighborhood  $F'' \subset \Xi(M(X))$  of  $\alpha_k''$  (choose  $F'' = B(\alpha_k'', \zeta_k'')$ ) there exist  $B(\alpha_k, \zeta_k) \subseteq \Xi^{-1}F''$  and  $n_{B(\alpha_k, \zeta_k), \alpha_k, \frac{\eta}{2}}^*$  satisfying

$$N(B(\alpha_k, \zeta_k); \delta^*, n, \epsilon^*) \geq \exp(n(h(T, \alpha_k) - \frac{\eta}{2})), \quad (3.4)$$

whence  $n \geq n_{B(\alpha_k, \zeta_k), \alpha_k, \frac{\eta}{2}}^*$  and  $\zeta_k, \zeta_k''$  will be determined later.

Choose three strictly decreasing sequences  $\{\zeta_k\}_k, \{\zeta_k''\}_k$  and  $\{\epsilon_k\}_k$ , such that

- (i)  $\lim_k \zeta_k = 0, \lim_k \zeta_k'' = 0$  and  $\lim_k \epsilon_k = 0$ ,
- (ii)  $\epsilon_1 < \epsilon^*$  and  $|\int \varphi d\alpha_k - \int \varphi d\mu| \leq \frac{\delta}{6} \quad \forall \mu \in B(\alpha_k, \zeta_k + 2\epsilon_k)$ .

From (3.4) we deduce the existence of  $n_k$  and a  $(\delta^*, n_k, \epsilon^*)$ -separated subset  $\Gamma_k \subseteq X_{n_k, B(\alpha_k, \zeta_k)} \subseteq X_{n_k, \Xi^{-1}B(\alpha_k'', \zeta_k'')}$  with

$$|\Gamma_k| \geq \exp\left(n_k \left(h(T, \alpha_k) - \frac{\eta}{2}\right)\right). \quad (3.5)$$

Assume that  $n_k$  satisfies

$$\delta^* n_k > 2g(n_k) + 1 \quad \text{and} \quad \frac{g(n_k)}{n_k} \leq \epsilon_k \quad (3.6)$$

The orbit-segments  $\{x, Tx, \dots, T^{n_k-1}x\}, x \in \Gamma_k$ , are the building-blocks for the construction of the points of  $G$ . By Proposition 2.2 and (3.6), we obtain

$$\begin{aligned} x \in \Gamma_k \text{ and } y \in B_{n_k}(g; x, \epsilon_k) \\ \Rightarrow \rho(\alpha_k, L_{n_k}y) \leq \zeta_k + 2\epsilon_k \\ \Rightarrow d'(\alpha_k'', \Xi L_{n_k}y) \leq V(\Xi, \zeta_k + 2\epsilon_k). \end{aligned} \quad (3.7)$$

Choose a strictly increasing sequence  $\{N_k\}$ , with  $N_k \in \mathbb{N}$ , such that

$$n_{k+1} \leq \zeta_k \sum_{j=1}^k n_j N_j \quad (3.8)$$

and

$$\sum_{j=1}^{k-1} n_j N_j \leq \zeta_k \sum_{j=1}^k n_j N_j. \quad (3.9)$$

Finally define the (stretched) sequences  $\{n'_j\}$ ,  $\{\epsilon'_j\}$  and  $\{\Gamma'_j\}$ , by setting

$$n'_j := n_k \quad \epsilon'_j := \epsilon_k \quad \Gamma'_j := \Gamma_k,$$

for  $j = N_1 + \cdots + N_{k-1} + q$  with  $1 \leq q \leq N_k$ .

$$G_k := \bigcap_{j=1}^k \left( \bigcup_{x_j \in \Gamma'_j} T^{-M_{j-1}} B_{n'_j}(g; x_j, \epsilon'_j) \right),$$

where  $M_j := \sum_{l=1}^j n'_l$ .  $G_k$  is a non-empty closed set. Label each set obtained by developing this formula by the branches of a labeled tree of height  $k$ . A branch is labeled by  $(x_1, \dots, x_k)$ , with  $x_j \in \Gamma'_j$ . Theorem 3.2 is proved by proving Lemma 3.2.  $\square$

**Lemma 3.2.** *Let  $\epsilon$  be such that  $4\epsilon = \epsilon^*$ , and let*

$$G := \bigcap_{k \geq 1} G_k.$$

(i) *Let  $x_j, y_j \in \Gamma'_j$  with  $x_j \neq y_j$ . If  $x \in B_{n'_j}(g; x_j, \epsilon'_j)$  and  $y \in B_{n'_j}(g; y_j, \epsilon'_j)$ , then*

$$\max\{d(T^m x, T^m y) : m = 0, \dots, n'_j - 1\} > 2\epsilon.$$

(ii)  *$G$  is a closed set, which is the disjoint union of non-empty closed sets  $G(x_1, x_2, \dots)$  labeled by  $(x_1, x_2, \dots)$  with  $x_j \in \Gamma'_j$ . Two different sequences label two different sets.*

(iii)  $G \subset G_{K'}^*$ .

(iv)  $P(G, \varphi) \geq h^*$ .

*Proof.* (i) and (ii) can be seen in [25] for details.

(iii) Define the stretched sequence  $\{\alpha'_m\}$  by  $\alpha'_m := \alpha_k$  if  $\sum_{j=1}^{k-1} n_j N_j + 1 \leq m \leq \sum_{j=1}^k n_j N_j$ .

The sequence  $\{\alpha'_m\}$  has the same limit-point set as the sequence  $\{\alpha_k\}$ ,  $\{\alpha''_m\}$  has the same limit-point set as the sequence  $\{\Xi \alpha_k\}$ . If

$$\lim_{n \rightarrow \infty} d'(\Xi L_n y, \alpha''_n) = 0,$$

then the two sequence  $\{\Xi L_n y\}$  and  $\{\alpha''_n\}$  have the same limit-point set. Because of (3.8) and the definition of  $\{\alpha''_m\}$ , it is sufficient to show that

$$\lim_{k \rightarrow \infty} d'(\Xi L_{M_k}(y), \alpha''_{M_k}) = 0.$$

Suppose that  $\sum_{l=1}^j n_l N_l < M_k \leq \sum_{l=1}^{j+1} n_l N_l$ ; hence  $\alpha'_{M_k} = \alpha_{j+1} \cdot M_k$  can be written as  $M_k = \sum_{l=1}^{j-1} n_l N_l + n_j N_j + q n_{j+1}$ , where  $1 \leq q \leq N_{j+1}$ .

Since

$$\begin{aligned} & \Xi(L_{\sum_{l=1}^{j-1} n_l N_l + n_j N_j + q n_{j+1}}(y)) \\ &= \Xi\left(\frac{1}{M_k} \left( \sum_{l=1}^{j-1} n_l N_l L_{\sum_{l=1}^{j-1} n_l N_l}(y) + \sum_{i=0}^{N_j-1} L_{n_j}(T^{\sum_{l=1}^{j-1} n_l N_l + i n_j} y) + \sum_{i=0}^{q-1} L_{n_{j+1}}(T^{\sum_{l=1}^j n_l N_l + i n_{j+1}} y) \right)\right) \\ &= \frac{\sum_{l=1}^{j-1} n_l N_l}{M_k} \Xi(L_{\sum_{l=1}^{j-1} n_l N_l}(y)) + \frac{\sum_{i=0}^{N_j-1} \Xi L_{n_j}(T^{\sum_{l=1}^{j-1} n_l N_l + i n_j} y)}{M_k} + \frac{\sum_{i=0}^{q-1} \Xi L_{n_{j+1}}(T^{\sum_{l=1}^j n_l N_l + i n_{j+1}} y)}{M_k} \end{aligned}$$

and

$$\alpha''_{M_k} = \frac{\sum_{l=1}^{j-1} n_l N_l}{M_k} \alpha''_{M_k} + \frac{\sum_{i=0}^{N_j-1} \alpha''_{M_k}}{M_k} + \frac{\sum_{i=0}^{q-1} \alpha''_{M_k}}{M_k},$$

we have

$$\begin{aligned} & d'(\Xi(L_{\sum_{l=1}^{j-1} n_l N_l + n_j N_j + q n_{j+1}}(y)), \alpha''_{M_k}) \\ & \leq \frac{\sum_{l=1}^{j-1} n_l N_l}{M_k} d'(\Xi(L_{\sum_{l=1}^{j-1} n_l N_l}(y)), \alpha''_{M_k}) + \frac{\sum_{i=0}^{N_j-1} d'(\Xi L_{n_j}(T^{\sum_{l=1}^{j-1} n_l N_l + i n_j} y), \Xi \alpha_{j+1})}{M_k} \\ & \quad + \frac{\sum_{i=0}^{q-1} d'(\Xi L_{n_{j+1}}(T^{\sum_{l=1}^j n_l N_l + i n_{j+1}} y), \Xi \alpha_{j+1})}{M_k} \\ & \leq \zeta_j V(\Xi, 1) + d'(\Xi \alpha_j, \Xi \alpha_{j+1}) + V(\Xi, \zeta_j + 2\epsilon_j) + V(\Xi, \zeta_{j+1} + 2\epsilon_{j+1}). \end{aligned}$$

Since  $\lim_j \zeta_j = 0, \lim_j \epsilon_j = 0 \Rightarrow \lim_j V(\Xi, \epsilon_j) = 0$  and  $\lim_j d'(\Xi \alpha_j, \Xi \alpha_{j+1}) = 0$  this proves (iii).

(iv) From the choice of  $\{N_k\}$  we can get  $\lim_{n \rightarrow \infty} \frac{M_n}{M_{n+1}} = 1$ , where  $M_j = n'_1 + \dots + n'_j$ . There exist  $n_k \in \mathbb{N}$  and a  $(\delta^*, n_k, \epsilon^*)$ -separated subset  $\Gamma_k$  of  $X_{n, B(\alpha_k, \zeta_k)}$  such that

$$\#\Gamma_k \geq \exp(n_k(h(T, \alpha_k) - \eta/2)).$$

And for any  ${}^kx \in \Gamma_k$ , we have  $L_n({}^kx) \in B(\alpha_k, \zeta_k)$ . So

$$|\int \varphi dL_n({}^kx) - \int \varphi d\alpha_k| = |\frac{1}{n}S_n\varphi({}^kx) - \int \varphi d\alpha_k| \leq \frac{\delta}{6}.$$

Thus

$$\begin{aligned} \#\Gamma_k &\geq \exp(n_k(h(T, \alpha_k) + \int \varphi d\alpha_k - \eta/2) - S_{n_k}\varphi({}^kx) - n_k\frac{\delta}{6}) \\ &\geq \exp(n_k h^* - S_{n_k}\varphi({}^kx) - n_k\frac{\delta}{6}). \end{aligned}$$

Since  $G$  is a compact set we can just consider finite covers  $\mathcal{C}$  of  $G$  with the property that if  $B_m(x, \epsilon) \in \mathcal{C}$ , then  $B_m(x, \epsilon) \cap G \neq \emptyset, \forall B_m(x, \epsilon) \in \mathcal{C}$ . For each  $\mathcal{C} \in \Gamma_n(G, \epsilon)$  we define the cover  $\mathcal{C}'$ , in which each ball  $B_m(x, \epsilon)$  is replaced by  $B_{M_p}(x, \epsilon)$  when  $M_p \leq m < M_{p+1}$ . Then

$$\begin{aligned} M(G, s, \varphi, n, \epsilon) &= \inf_{\mathcal{C} \in \Gamma_n(G, \epsilon)} \sum_{B_m(x, \epsilon) \in \mathcal{C}} \exp(-sm + \sup_{y \in B_m(x, \epsilon)} S_m\varphi(y)) \\ &\geq \inf_{\mathcal{C} \in \Gamma_n(G, \epsilon)} \sum_{\substack{B_{M_p}(x, \epsilon) \in \mathcal{C}', \\ z \in B_m(x, \epsilon) \cap B_{M_p}(x, \epsilon) \cap G}} \exp(-sm + S_m\varphi(z)) \end{aligned}$$

Consider a specific  $\mathcal{C}'$  and let  $m$  be the largest value of  $p$  such that there exists  $B_{M_p}(x, \epsilon) \in \mathcal{C}'$ .

Set

$$\mathcal{W}_k := \prod_{i=1}^k \Gamma'_k, \overline{\mathcal{W}}_m := \bigcup_{k=1}^m \mathcal{W}_k.$$

Each  $z \in B_{M_p}(x, \epsilon) \cap G$  corresponds to a point in  $\mathcal{W}_p$ . Lemma 3.2 (i) implies that this point is uniquely defined. For  $1 \leq j \leq k$ , the word  $v = (v_1, \dots, v_j) \in \mathcal{W}_j$  is a prefix of  $w = (w_1, \dots, w_k) \in \mathcal{W}_k$ , if  $v_i = w_i, i = 1, \dots, j$ . Note that each  $w \in \mathcal{W}_k$  is the prefix of exactly  $|\mathcal{W}_m|/|\mathcal{W}_k|$  words of  $\mathcal{W}_m$ . If  $\mathcal{W} \subset \overline{\mathcal{W}}_m$  contains a prefix of each word of  $\mathcal{W}_m$ , then

$$\sum_{k=1}^m |\mathcal{W} \cap \mathcal{W}_k| |\mathcal{W}_m| / |\mathcal{W}_k| \geq |\mathcal{W}_m|.$$

So if  $\mathcal{W}$  contains a prefix of each word of  $\mathcal{W}_m$ , then

$$\sum_{k=1}^m |\mathcal{W} \cap \mathcal{W}_k| / |\mathcal{W}_k| \geq 1.$$

Since  $\mathcal{C}'$  is a cover, each point of  $\mathcal{W}_m$  has a prefix associated with some  $B_{M_p}(x, \epsilon) \in \mathcal{C}'$ . Hence,

$$|\mathcal{W}_p| \geq \exp[M_p h^* - \sum_{i=1}^p (S_{n'_i} \varphi(i'x) + n'_i \delta/6)],$$

where  $i'x \in \Gamma'_i$ . So

$$\sum_{B_{M_p}(x, \epsilon) \in \mathcal{C}'} \exp[-M_p h^* + \sum_{i=1}^p (S_{n'_i} \varphi(i'x) + n'_i \delta/6)] \geq 1.$$

Next, we want to prove  $M_p h^* - \sum_{i=1}^p (S_{n'_i} \varphi(i'x) + n'_i \delta/6) - sm + S_m \varphi(z) = m(h^* - s) + \sum_{i=1}^p (S_{n'_i} \varphi(T^{M_{i-1}} z) - S_{n'_i} \varphi(i'x) - n'_i \delta/6) + S_{m-M_p} \varphi(T^{M_p} z) - (m - M_p) h^* > 0$ . Since  $z \in G$ , by the construction of  $G$  we know there exists a close subset

$$G(x_1, x_2, \dots) = \bigcap_{i=0}^{\infty} T^{-M_{j-1}} B_{n'_j}(g; x_j, \epsilon'_j),$$

such that  $T^{M_{j-1}} z \in B_{n'_j}(g; x_j, \epsilon'_j)$ .

By (3.7) and  $i'x \in \Gamma'_i$  we get  $L_{n'_i}(T^{M_{i-1}} z) \in B(\alpha'_i, \zeta'_i + 2\epsilon'_i)$  and  $L_{n'_i}(i'x) \in B(\alpha'_i, \zeta'_i)$ . Thus,

$$\left| \int \varphi dL_{n'_i}(T^{M_{i-1}} z) - \int \varphi dL_{n'_i}(i'x) \right| n'_i = \left| S_{n'_i} \varphi(T^{M_{i-1}} z) - S_{n'_i} \varphi(i'x) \right| \leq n'_i \delta/2.$$

So,

$$\begin{aligned} & M_p h^* - \sum_{i=1}^p (S_{n'_i} \varphi(i'x) + n'_i \delta/6) - sm + S_m \varphi(z) \\ & \geq m(h^* - s) - \sum_{i=1}^p 2n'_i \delta/3 - n'_{p+1}(\|\varphi\| + h^*) \\ & \geq 2\delta M_p - M_p \delta - n'_{p+1}(\|\varphi\| + h^*) \\ & \geq M_p \delta - n'_{p+1}(\|\varphi\| + h^*). \end{aligned}$$

Since  $\lim_{p \rightarrow \infty} \frac{n'_{p+1}}{M_p} = 0$ , it is possible to choose sufficient large  $p$  such that  $M_p \delta - n'_{p+1}(\|\varphi\| + h^*) > 0$ . Then

$$\sum_{B_m(x, \epsilon) \in \mathcal{C}} \exp(-sm + \sup_{y \in B_m(x, \epsilon)} S_m \varphi(y)) \geq \sum_{B_{M_p}(x, \epsilon) \in \mathcal{C}'} \exp[-M_p h^* + \sum_{i=1}^p (S_{n'_i} \varphi(i'x) + n'_i \delta/6)].$$

It implies  $M(G, s, \varphi, n, \epsilon) \geq 1$ , i.e.,  $s \leq P(G, \varphi, \epsilon)$ . Letting  $s \rightarrow h^*$ , we complete the proof of Lemma 3.2.  $\square$

**Remark 3.3.** The quintuple  $(X, T, Y, \Xi, L_n)$  satisfying  $g$ -almost product property and the uniform separated condition means:

- (a)  $X$  is a compact metric space,  $T : X \rightarrow X$  is a continuous map satisfying  $g$ -almost product property and the uniform separated condition.
- (b)  $Y$  a vector space,  $\Xi : M(X) \rightarrow Y$  is a continuous and affine map.
- (c)  $L_n x : X \rightarrow M(X)$ , where  $L_n x = \sum_{i=0}^{n-1} \delta_{T^i x}$ .

For  $y \in Y$ , set

$$\Delta(y) = \{x \in X : \{y\} = A(\Xi L_n x)\}, \tilde{\Delta}(y) = \{x \in X : y \in A(\Xi L_n x)\}.$$

It is easy to get the following corollary by the above two theorems.

**Corollary 3.1.** For any  $\varphi \in C(X, \mathbb{R})$ , if  $(X, T, Y, \Xi, L_n)$  satisfies  $g$ -almost product property and the uniform separated condition, then

$$P(\Delta(y), \varphi) = P(\tilde{\Delta}(y), \varphi).$$

## 4 Proof of Theorems

This section is aim to prove the theorems.

*Proof of Theorem 1.1*

(1) It can be obtained by proposition 2.6.

(2)  $C \subset Y$  is a compact and connected subset of  $\Xi(M(X, T))$ , then  $P(\Delta_{equ}(C), \varphi) = P(G_C^*, \varphi) = \inf_{y \in C} \Lambda(y, \varphi)$ .

*Proof of Theorem 1.2*

(1) It is obvious.

(2) We prove it by presenting several lemmas.

**Lemma 4.1.** Let  $(X, T, Y, \Xi, L_n)$  satisfy  $g$ -almost product property and the uniform separated condition. If  $C \subset Y$  and  $\varphi \in C(X, \mathbb{R})$ , then

$$\inf_{y \in C} \Lambda(y, \varphi) = \inf_{y \in co(C)} \Lambda(y, \varphi),$$

where  $co(C)$  is convex hull of  $C$ .

*Proof.* The direction  $\geq$  is obvious. As to the other direction, for any  $y \in co(C)$ , fix  $\epsilon > 0$ . Since  $y \in co(C)$ , there exist  $y_1, y_2, \dots, y_n \in C$  and  $\lambda_1, \dots, \lambda_n \geq 0$  with  $\sum_i \lambda_i = 1$  such that  $\sum_i \lambda_i y_i = y$ . For each  $y_i$ , choose  $\mu_i \in M(X, T)$  s.t.  $\Xi \mu_i = y_i$  and  $h(T, \mu_i) +$



$\int \varphi d\mu_i \geq \Lambda(y_i, \varphi) - \epsilon$ . Since the entropy function is affine and  $\sum_i \lambda_i \mu_i \in M(X, T)$  satisfies  $\Xi(\sum_i \lambda_i \mu_i) = \sum_i \lambda_i \Xi \mu_i = \sum_i \lambda_i y_i = y$ , we get

$$\begin{aligned}
\Lambda(y, \varphi) &= \sup_{\mu \in M(X, T), \Xi \mu = y} \{h(T, \mu) + \int \varphi d\mu\} \\
&\geq h(T, \sum_i \lambda_i \mu_i) + \int \varphi d(\sum_i \lambda_i \mu_i) \\
&= \sum_i \lambda_i (h(T, \mu_i) + \int \varphi d\mu_i) \\
&\geq \sum_i \lambda_i \Lambda(y_i, \varphi) - \epsilon \\
&\geq \inf_{y \in C} \Lambda(y, \varphi) - \epsilon.
\end{aligned}$$

Thus,

$$\inf_{y \in C} \Lambda(y, \varphi) \leq \inf_{y \in \text{co}(C)} \Lambda(y, \varphi).$$

□

**Lemma 4.2.** *Let  $(X, T, Y, \Xi, L_n)$  satisfy g-almost product property and the uniform separated condition. If  $C \subset Y$  and  $\varphi \in C(X, \mathbb{R})$ , then*

$$\inf_{y \in C} \Lambda(y, \varphi) = \inf_{y \in \overline{C}} \Lambda(y, \varphi).$$

*Proof.*  $\geq$  is obvious. The other direction follows from the fact that  $y \rightarrow \Lambda(y, \varphi)$  is upper semi-continuous. C. Pfister and W. Sullivan [25] proved that the entropy map on  $M(X, T), \mu \rightarrow h(T, \mu)$ , is upper semi-continuous under the g-almost product property and uniformly separation property.  $\forall \gamma > 0, \forall y \in \overline{C}, \exists \{y_n\} \subseteq C$ , s.t.  $y_n \rightarrow y$ , as  $n \rightarrow \infty$  and there exists  $\mu_n \in M(X, T) \cap \Xi^{-1} y_n$ , s.t.  $\Lambda(y_n, \varphi) \leq h(T, \mu_n) + \int \varphi d\mu_n + \gamma$ . Assume that  $\mu$  is a limit-point of  $\{\mu_n\}$ , then  $\Xi \mu = y$ . So

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \Lambda(y_n, \varphi) &\leq \limsup_{n \rightarrow \infty} h(T, \mu_n) + \int \varphi d\mu_n + \gamma \\
&\leq h(T, \mu) + \int \varphi d\mu + 3\gamma \\
&\leq \Lambda(y, \varphi) + 3\gamma.
\end{aligned}$$

The conclusion of Lemma 4.2 follows.

Now, continue the proof of (2). It suffices to show for any nonempty  $C \subset \Xi(M(X, T))$ ,

$$P(\Delta_{equ}(\overline{co(C)}), \varphi) = P(\Delta_{sub}(C), \varphi).$$

Since  $\Delta_{equ}(\overline{co(C)}) = \{x \in X \mid A(\Xi L_n x) = \overline{co(C)}\} \subset \{x \in X \mid C \subset A(\Xi L_n x)\} = \Delta_{sub}(C)$ , it is obvious that  $P(\Delta_{sub}(C), \varphi) \geq P(\Delta_{equ}(\overline{co(C)}), \varphi)$ . On the other hand, by Corollary 3.1, if  $\Delta(y) \neq \emptyset$ , then  $P(\Delta(y), \varphi) = P(\tilde{\Delta}(y), \varphi)$ . So for any  $y \in C$ ,

$$\begin{aligned} P(\Delta_{sub}(C), \varphi) &\leq P(\{x \in X \mid \{y\} \subset A(\Xi L_n x)\}, \varphi) \\ &= P(\tilde{\Delta}(y), \varphi) \\ &= P(\Delta(y), \varphi) \\ &= \Lambda(y, \varphi). \end{aligned}$$

Hence,

$$\begin{aligned} P(\Delta_{sub}(C), \varphi) &\leq \inf_{y \in C} \Lambda(y, \varphi) \\ &= \inf_{y \in \overline{co(C)}} \Lambda(y, \varphi) \\ &= P(\Delta_{equ}(\overline{co(C)}), \varphi). \end{aligned}$$

So,

$$P(\Delta_{sub}(C), \varphi) = \inf_{y \in C} \sup_{\substack{\mu \in M(X, T) \\ \Xi \mu = y}} \{h(T, \mu) + \int \varphi d\mu\} = \inf_{y \in C} \Lambda(y, \varphi).$$

□

*Proof of Theorem 1.3*

First, we show the following proposition.

**Proposition 4.1.**  *$(X, T, Y, \Xi, L_n)$  as before and  $\varphi \in C(X, \mathbb{R})$ . If  $C \subset Y$ , then*

$$P(\Delta_{sup}(C), \varphi) = P(\Delta_{sup}(C \cap \Xi(M(X, T))), \varphi) = \sup_{y \in C} \Lambda(y, \varphi).$$

*Proof.*  $P(\Delta_{sup}(C), \varphi) \leq P({}^C G^*, \varphi) \leq \sup_{y \in C} \Lambda(y, \varphi)$ .

On the other hand,  $\forall \epsilon > 0, \exists y' \in C \cap \Xi(M(X, T))$ , s.t.  $\sup_{y \in C} \Lambda(y, \varphi) \leq \Lambda(y', \varphi) + \epsilon$ , and  $\Lambda(y', \varphi) = P(G_{\{y'\}}^*, \varphi) \leq P(\Delta_{sup}(C), \varphi)$ .

So,

$$P(\Delta_{sup}(C), \varphi) + \epsilon \geq \sup_{y \in C} \Lambda(y, \varphi).$$

Thus,

$$P(\Delta_{sup}(C), \varphi) = \sup_{y \in C} \Lambda(y, \varphi).$$

□

Now, We continue the proof of Theorem 1.3.

(1) It comes from Proposition 4.1.

(2) Given  $S_1 \subseteq Q \subseteq S_2$ ,  $Q \subseteq \Xi(M(X, T))$  is compact and connected.

Since  $\Delta_{equ}(Q) = \{x \in X | A(\Xi L_n x) = Q\} \subset \{S_1 \subset A(\Xi L_n x) \subset S_2\}$ , we get

$$P(\Delta(S_1, S_2), \varphi) \geq P(\Delta_{equ}(Q), \varphi) = \inf_{x \in Q} \Lambda(x, \varphi).$$

Since  $Q$  is arbitrary, we obtain

$$P(\Delta(S_1, S_2), \varphi) \geq \sup_{\substack{Q \subseteq \Xi(M(X, T)) \\ S_1 \subseteq Q \subseteq S_2 \\ Q \text{ is compact and connected}}} \inf_{x \in Q} \Lambda(x, \varphi).$$

As to the other inequality, observe

$$\Delta(S_1, S_2) \subset \Delta(S_1, Y) = \Delta_{sub}(S_1).$$

(3) It is obvious.

*Proof of Theorem 1.4*

(1) It follows from Proposition 4.1.

(2) Combining the fact  $S_1 \subseteq \overline{co}(S_1)$  and  $\overline{co}(S_1)$  is a compact and connected subset of  $S_2$ , and Theorem 1.3, we obtain

$$\begin{aligned} \inf_{y \in S_1} \Lambda(y, \varphi) &= \inf_{y \in \overline{co}(S_1)} \Lambda(y, \varphi) \\ &\leq \sup_{\substack{S_1 \subseteq Q \subseteq S_2 \\ Q \text{ is compact and connected}}} \inf_{y \in Q} \Lambda(y, \varphi) \\ &\leq P(\Delta(S_1, S_2), \varphi) \\ &\leq \inf_{y \in S_1} \Lambda(y, \varphi). \end{aligned}$$

(3) It is obvious.

## 5 Some applications

In the section, firstly, we present some spectra induced by different deformations. Secondly, we use BS-dimension to describe some level sets. Thirdly, the relative multifractal spectrum of ergodic averages are discussed. At last, symbolic space and iterated function systems are investigated.

## 5.1 Some spectra

Different spectra are induced by different deformations  $(Y, \Xi)[15]$ .

- The spectrum of the historic set of ergodic averages. Let  $\varphi : X \rightarrow \mathbb{R}$  be continuous and define  $\Xi : M(X) \rightarrow \mathbb{R}$  by  $\Xi : \mu \mapsto \int \varphi d\mu$ . In this case we obtain for  $S_1, S_2 \subset \mathbb{R}$ ,

$$\Delta(S_1, S_2) = \left\{ x \in X : S_1 \subset A \left( \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k x) \right) \subset S_2 \right\}.$$

- The spectrum of the historic set of empirical measures. Define  $\Xi : M(X) \rightarrow M(X)$  by

$$\Xi : \mu \rightarrow \mu.$$

In this case we obtain for  $S_1, S_2 \subset \mathbb{R}$ ,

$$\Delta(S_1, S_2) = \{x \in X : S_1 \subset A(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k x}) \subset S_2\}.$$

- The spectrum of the historic set of local Lyapunov exponents. Let  $X$  be a differentiable manifold and  $T : X \rightarrow X$  be a  $C^1$  map. The local Lyapunov exponents of  $T$  at the point  $x$  is defined by  $\chi(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |(DT^n)(x)|$ . Define  $\Xi : M(X) \rightarrow \mathbb{R}$  by

$$\mu \rightarrow \int DT d\mu.$$

In this case we obtain for  $S_1, S_2 \subseteq \mathbb{R}$ ,

$$\Delta(S_1, S_2) = \{x \in X : S_1 \subset A(\frac{1}{n} \log |(DT^n)(x)|) \subset S_2\}.$$

- The mixed spectrum of the historic set of ergodic averages of arbitrary families of continuous functions. Assume that the family of maps  $(M(X) \rightarrow \mathbb{R} : \mu \mapsto \int \varphi_i d\mu)_{i \in I}$  is totally bounded. Define  $\Xi : M(X) \rightarrow l^\infty(I)$  by

$$\Xi : \mu \mapsto (\int \varphi_i d\mu)_{i \in I}.$$

In this case we obtain for  $S_1, S_2 \subset l^\infty(I)$ ,

$$\Delta(S_1, S_2) = \{x \in X : S_1 \subset A((\frac{1}{n} \sum_{k=0}^{n-1} \varphi_i(T^k x))_{i \in I}) \subset S_2\}$$

We only consider some (not all) spectrum above and obtain several corollaries as examples. It is easy to get Corollary 5.1 and Corollary 5.2. So we omit the proof.

**Corollary 5.1.**  *$(X, T, L_n)$  as before. Let  $Y = \mathbb{R}$  and  $\phi_j : X \rightarrow \mathbb{R}$  be a family of continuous functions. Assume the family of maps  $(\Xi_j : M(X) \rightarrow \mathbb{R} : \mu \mapsto \int \phi_j d\mu)_{j \in I}$  is totally bounded. Fix  $S_1, S_2 \subset l^\infty(I)$ ,  $\psi \in C(X, \mathbb{R})$ .*

1. *If  $S_1 = \emptyset$  and  $S_2$  is closed and convex, then*

$$P(\{x \in X : A((\frac{1}{n} \sum_{k=0}^{n-1} \phi_j(T^k x))_{j \in I})\} \subset S_2, \psi) = \sup_{x \in S_2} \Lambda(x, \psi).$$

2. *If  $S_1 \neq \emptyset$  and  $\overline{co}(S_1)$  is contained in a connected component of  $S_2$ , then*

$$P(\{x \in X : S_1 \subset A((\frac{1}{n} \sum_{k=0}^{n-1} \phi_j(T^k x))_{j \in I})\}, \psi) = \inf_{x \in S_1} \Lambda(x, \psi).$$

3. *If  $S_1 \neq \emptyset$  and  $\overline{co}(S_1)$  is not contained in a connected component of  $S_2$ , then*

$$\{x \in X : S_1 \subset A((\frac{1}{n} \sum_{k=0}^{n-1} \phi_j(T^k x))_{j \in I})\} = \emptyset.$$

**Corollary 5.2.**  *$(X, T, L_n)$  as before. Let  $(Y_i, \Xi_i)_i$  be (a possible uncountable) family of deformations and assume that  $Y_i$  is a normed vector space and that  $\Xi_i : M(X) \rightarrow Y_i$  is affine and continuous. Define the vector spaces  $\times_i Y_i$  and  $[\times_i Y_i]^\infty$  by*

$$\times_i Y_i = \{(y_i)_i : y_i \in Y_i \forall i\},$$

$$[\times_i Y_i]^\infty = \{(y_i)_i \in \times_i Y_i : \sup_i \|y_i\| < \infty\},$$

*and equip  $[\times_i Y_i]^\infty$  with the norm  $\|(y_i)_i\| = \sup_i \|y_i\|$ . Assume  $\sup_{\mu \in M(X), i} \|\Xi_i \mu\| < \infty$  and the map*

$$M(X) \rightarrow [\times_i Y_i]^\infty : \mu \mapsto (\Xi_i \mu)_i$$

*is continuous,  $\Xi = (\Xi_i)_{i \in I}$ . Fix  $S_1, S_2 \subset [\times_i Y_i]^\infty$ ,  $\psi \in C(X, \mathbb{R})$*

1. *If  $S_1 = \emptyset$  and  $S_2$  is closed and convex, then*

$$P(\{x \in X : S_1 \subset A((\Xi_j L_n x)_{j \in I}) \subset S_2\}, \psi) = \sup_{x \in S_2} \sup_{\substack{\mu \in M(X, T) \\ (\Xi_j \mu)_{j \in I} = x}} \left\{ h(T, \mu) + \int \psi d\mu \right\}.$$

2. If  $S_1 \neq \emptyset$  and  $\overline{co}(S_1)$  is contained in a connected component of  $S_2$ , then

$$P(\{x \in X : S_1 \subset A((\Xi_j L_n x)_{j \in I}) \subset S_2\}, \psi) = \inf_{x \in S_1} \sup_{\substack{\mu \in M(X, T) \\ (\Xi_j \mu)_{j \in I} = x}} \left\{ h(T, \mu) + \int \psi d\mu \right\}.$$

3. If  $S_1 \neq \emptyset$  and  $\overline{co}(S_1)$  is not contained in a connected component of  $S_2$ , then

$$\{x \in X : S_1 \subset A((\Xi_j L_n x)_{j \in I}) \subset S_2\} = \emptyset.$$

Next, we use dimension theory to discuss  $\Delta_{equ}(\cdot)$ ,  $\Delta_{sub}(\cdot)$  and so on.

Let  $\psi : X \rightarrow \mathbb{R}$  be a strictly positive continuous function. For each set  $Z \subset X$  and each number  $t \in \mathbb{R}$ , define

$$N(Z, t, \psi, n, \epsilon) := \inf_{\mathcal{C} \in \mathcal{G}_n(Z, \epsilon)} \left\{ \sum_{B_m(x, \epsilon) \in \mathcal{C}} \exp(-t \sup_{y \in B_m(x, \epsilon)} S_m \psi(y)) \right\},$$

where  $\mathcal{G}_n(Z, \epsilon)$  is the collection of all finite or countable covers of  $Z$  by sets of the form  $B_m(x, \epsilon)$ , with  $m \geq n$ .

$$N(Z, t, \psi, \epsilon) = \lim_{N \rightarrow \infty} N(Z, t, \psi, n, \epsilon),$$

Set

$$BS(Z, \psi, \epsilon) = \inf\{t : N(Z, t, \psi, \epsilon) = 0\} = \sup\{t : N(Z, t, \psi, \epsilon) = \infty\}.$$

Let  $BS(Z, \psi) = \lim_{\epsilon \rightarrow 0} BS(Z, \psi, \epsilon)$ , and we call it the BS-dimension of  $Z$ . This notation was introduced by Barreira and Schmeling [4].

By the definition of topological pressure and BS-dimension, we can get that for any set  $Z \subset X$ , the BS-dimension of  $Z$  is a unique root of Bowen's equation  $P(Z, -s\psi) = 0$ , i.e.  $s = BS(Z, \psi)$ .

The following corollaries are easy to obtain from above theorems.

**Corollary 5.3.**  $(X, T, \Xi, L_n, Y)$  satisfies  $g$ -almost product property and the uniform separation property and  $\varphi \in C(X, \mathbb{R}^+)$ . If

1.  $C \subset Y$  is not a compact and connected subset of  $\Xi(M(X, T))$ , then

$$\{x \in X : A(\Xi L_n x) = C\} = \emptyset,$$

2.  $C \subset Y$  is a compact and connected subset of  $\Xi(M(X, T))$ , then

$$BS(\Delta_{equ}(C), \varphi) = \inf_{y \in C} \sup_{\substack{\mu \in M(X, T) \\ \Xi \mu = y}} \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} \right\}.$$

**Corollary 5.4.**  $(X, T, \Xi, L_n, Y)$  as before and  $\varphi \in C(X, \mathbb{R}^+)$ . If

1.  $C \subset Y$  is not a subset of  $\Xi(M(X, T))$ , then

$$\{x \in X : C \subset A(\Xi L_n x)\} = \emptyset.$$

2.  $C \subset Y$  is a subset of  $\Xi(M(X, T))$ , then

$$BS(\Delta_{sub}(C), \varphi) = \inf_{y \in C} \sup_{\substack{\mu \in M(X, T) \\ \Xi \mu = y}} \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} \right\}.$$

**Corollary 5.5.**  $(X, T, \Xi, L_n, Y)$  as before and  $\varphi \in C(X, \mathbb{R}^+)$ , fix  $S_1 \subset \Xi(M(X, T))$ ,  $S_2 \subset Y$ , if

1.  $S_1 = \emptyset$ , then

$$BS(\Delta(S_1, S_2), \varphi) = \sup_{x \in S_2} \sup_{\substack{\Xi \mu = x \\ \mu \in M(X, T)}} \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} \right\}.$$

2.  $S_1 \neq \emptyset$  and  $S_1$  is contained in a connected component of  $S_2$ , then

$$\begin{aligned} & \sup_{\substack{S_1 \subseteq Q \subseteq S_2 \\ Q \subseteq \Xi(M(X, T)) \text{ is compact and connected}}} \inf_{x \in Q} \sup_{\substack{\Xi \mu = x \\ \mu \in M(X, T)}} \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} \right\} \\ & \leq BS(\Delta(S_1, S_2), \varphi) \leq \inf_{x \in S_1} \sup_{\substack{\Xi \mu = x \\ \mu \in M(X, T)}} \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} \right\}. \end{aligned}$$

3.  $S_1 \neq \emptyset$  and  $S_1$  is not contained in a connected component of  $S_2$ , then

$$\{x \in X : S_1 \subset A(\Xi L_n x) \subset S_2\} = \emptyset.$$

**Corollary 5.6.**  $(X, T, \Xi, L_n, Y)$  as before and  $\varphi \in C(X, \mathbb{R}^+)$ , fix  $S_1 \subset \Xi(M(X, T))$ ,  $S_2 \subseteq Y$ .

1. If  $S_1 = \emptyset$ , then

$$BS(\Delta(S_1, S_2), \varphi) = \sup_{x \in S_2} \sup_{\substack{\Xi \mu = x \\ \mu \in M(X, T)}} \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} \right\}.$$

2. If  $S_1 \neq \emptyset$  and  $\overline{\text{co}}(S_1)$  the closed convex hull of  $S_1$  is contained in a connected component of  $S_2$ , then

$$BS(\Delta(S_1, S_2), \varphi) = \inf_{x \in S_1} \sup_{\substack{\Xi \mu = x \\ \mu \in M(X, T)}} \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} \right\}.$$

3. If  $S_1 \neq \emptyset$  and  $S_1$  is not contained in a connected component of  $S_2$ , then

$$\{x \in X : S_1 \subset A(\Xi L_n x) \subset S_2\} = \emptyset.$$

**Corollary 5.7.**  $(X, T, L_n)$  as before. Let  $Y = \mathbb{R}$  and  $\phi_j : X \rightarrow \mathbb{R}$  be a family of continuous functions. Assume the family of maps  $(\Xi_j : M(X) \rightarrow \mathbb{R} : \mu \mapsto \int \phi_j d\mu)_{j \in I}$  is totally bounded,  $\Xi = (\Xi_i)_{i \in I}$ . Fix  $S_1, S_2 \subset l^\infty(I)$ ,  $\varphi \in C(X, \mathbb{R}^+)$ .

1. If  $S_1 = \emptyset$  and  $S_2$  is closed and convex, then

$$BS \left( \left\{ x \in X : A \left( \left( \frac{1}{n} \sum_{k=0}^{n-1} \phi_j(T^k x) \right)_{j \in I} \right) \right\} \subset S_2, \varphi \right) = \sup_{x \in S_2} \sup_{\substack{\Xi \mu = x \\ \mu \in M(X, T)}} \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} \right\}.$$

2. If  $S_1 \neq \emptyset$  and  $\overline{\text{co}}(S_1)$  is contained in a connected component of  $S_2$ , then

$$BS \left( \left\{ x \in X : S_1 \subset A \left( \left( \frac{1}{n} \sum_{k=0}^{n-1} \phi_j(T^k x) \right)_{j \in I} \right) \right\}, \varphi \right) = \inf_{x \in S_1} \sup_{\substack{\Xi \mu = x \\ \mu \in M(X, T)}} \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} \right\}.$$

3. If  $S_1 \neq \emptyset$  and  $\overline{\text{co}}(S_1)$  is not contained in a connected component of  $S_2$ , then

$$\left\{ x \in X : S_1 \subset A \left( \left( \frac{1}{n} \sum_{k=0}^{n-1} \phi_j(T^k x) \right)_{j \in I} \right) \right\} = \emptyset.$$

**Corollary 5.8.**  $(X, T, L_n)$  as before. Let  $(Y_i, \Xi_i)_i$  be (a possible uncountable) family of deformations and assume that  $Y_i$  is a normed vector space and that  $\Xi_i : M(X) \rightarrow Y_i$  is affine and continuous. Define the vector spaces  $\times_i Y_i$  and  $[\times_i Y_i]^\infty$  by

$$\times_i Y_i = \{(y_i)_i | y_i \in Y_i \forall i\},$$

$$[\times_i Y_i]^\infty = \{(y_i)_i \in \times_i Y_i | \sup_i \|y_i\| < \infty\},$$

and equip  $[\times_i Y_i]^\infty$  with the norm  $\|(y_i)_i\| = \sup_i \|y_i\|$ . Assume  $\sup_{\mu \in M(X), i} \|\Xi_i \mu\| < \infty$  and the map

$$M(X) \rightarrow [\times_i Y_i]^\infty : \mu \mapsto (\Xi_i \mu)_i$$

is continuous. Fix  $S_1, S_2 \subset [\times_i Y_i]^\infty$ ,  $\varphi \in C(X, \mathbb{R}^+)$



1. If  $S_1 = \emptyset$  and  $S_2$  is closed and convex, then

$$BS(\{x \in X : S_1 \subset A((\Xi_j L_n x)_{j \in I}) \subset S_2\}, \varphi) = \sup_{x \in S_2} \sup_{\substack{\mu \in M(X, T) \\ (\Xi_j \mu)_{j \in I} = x}} \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} \right\}.$$

2. If  $S_1 \neq \emptyset$  and  $\overline{\text{co}}(S_1)$  is contained in a connected component of  $S_2$ , then

$$BS(\{x \in X : S_1 \subset A((\Xi_j L_n x)_{j \in I}) \subset S_2\}, \varphi) = \inf_{x \in S_1} \sup_{\substack{\mu \in M(X, T) \\ (\Xi_j \mu)_{j \in I} = x}} \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} \right\}.$$

3. If  $S_1 \neq \emptyset$  and  $\overline{\text{co}}(S_1)$  is not contained in a connected component of  $S_2$ , then

$$\{x \in X : S_1 \subset A((\Xi_j L_n x)_{j \in I}) \subset S_2\} = \emptyset.$$

## 5.2 The relative multifractal spectrum of ergodic averages

The relative multifractal spectrum of ergodic averages. Let  $f, g \in C(X, \mathbb{R})$  with  $g(x) \neq 0$  for all  $x \in X$  and  $C \subseteq \mathbb{R}$ . Define  $\Xi : M(X) \rightarrow \mathbb{R}$  by  $\Xi : \mu \mapsto \frac{\int f d\mu}{\int g d\mu}$ . Here remark that  $\Xi$  is continuous but not affine.

**Corollary 5.9.** *( $X, T, L_n$ ) as before. Let  $f_1, g_1, \dots, f_m, g_m$  be continuous functions  $f_i, g_i : X \rightarrow \mathbb{R}$  with  $g_i(x) \neq 0$  for all  $x \in X, i = 1, \dots, m$  and  $\int g_i d\mu \neq 0$ , for all  $\mu \in M(X, T), i = 1, \dots, m$ . If  $C \subseteq \mathbb{R}^m$  is closed and convex,  $\psi \in C(X, \mathbb{R})$ , then*

$$\begin{aligned} & P \left( \left\{ x \in X : A \left( \left( \frac{\sum_{k=0}^{n-1} f_j(T^k x)}{\sum_{k=0}^{n-1} g_j(T^k x)} \right)_{j \in \{1, 2, \dots, m\}} \right) \subseteq C \right\}, \psi \right) \\ &= \sup \left\{ h(T, \mu) + \int \psi d\mu : \mu \in M(X, T), \left( \frac{\int f_i d\mu}{\int g_i d\mu} \right)_{i \in \{1, \dots, m\}} \in C \right\}. \end{aligned}$$

*Proof.* Since the map  $\Xi : \mu \mapsto \left( \frac{\int f_i d\mu}{\int g_i d\mu} \right)_{i=1, \dots, m}$  is continuous, we have

$$\begin{aligned} & \{x \in X : A(\Xi L_n(x)) \subseteq C\} \\ &= \{x \in X : \Xi A(L_n(x)) \subseteq C\} \\ &= \{x \in X : A(L_n(x)) \subseteq \Xi^{-1}C\} \\ &\subseteq \{x \in X : A(L_n(x)) \cap \Xi^{-1}C \neq \emptyset\} \\ &= \{x \in X : A(L_n(x)) \cap (\Xi^{-1}C \cap M(X, T)) \neq \emptyset\}. \end{aligned}$$

It follows from Proposition 3.1 (i) that

$$\begin{aligned}
& P \left( \left\{ x \in X : A \left( \left( \frac{\sum_{k=0}^{n-1} f_j(T^k x)}{\sum_{k=0}^{n-1} g_j(T^k x)} \right)_{j \in \{1,2,\dots,m\}} \right) \subseteq C \right\}, \psi \right) \\
& \leq \sup \left\{ h(T, \mu) + \int \psi d\mu : \mu \in M(X, T), \left( \frac{\int f_i d\mu}{\int g_i d\mu} \right)_{i \in \{1,\dots,m\}} \in C \right\}.
\end{aligned}$$

To the opposite inequality, we prove the case  $m = 1$  as example. For any  $\alpha \in C$ ,

$$\begin{aligned}
& \left\{ x \in X : A \left( \frac{1}{n} \sum_{k=0}^{n-1} (f_1(T^k x) - \alpha g_1(T^k x)) \right) = 0 \right\} \\
& \subseteq \left\{ x \in X : A \left( \frac{\sum_{k=0}^{n-1} f_1(T^k x)}{\sum_{k=0}^{n-1} g_1(T^k x)} \right) \subseteq C \right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \sup \left\{ h(T, \mu) + \int \psi d\mu : \frac{\int f_1 d\mu}{\int g_1 d\mu} = \alpha \in C, \mu \in M(X, T) \right\} \\
& = \sup \left\{ h(T, \mu) + \int \psi d\mu : \int f_1 - \alpha g_1 d\mu = 0, \alpha \in C, \mu \in M(X, T) \right\} \\
& = \sup_{\alpha \in C} P \left( \left\{ x \in X : A \left( \frac{1}{n} \sum_{k=0}^{n-1} (f_1(T^k x) - \alpha g_1(T^k x)) \right) = 0 \right\}, \psi \right) \\
& \leq P \left( \left\{ x \in X : A \left( \frac{\sum_{k=0}^{n-1} f_1(T^k x)}{\sum_{k=0}^{n-1} g_1(T^k x)} \right) \subseteq C \right\}, \psi \right).
\end{aligned}$$

Since the case  $m > 1$  is similar to  $m = 1$ , the proof is omitted.  $\square$

**Corollary 5.10.** *( $X, T, L_n$ ) as before. Let  $f_1, g_1, \dots, f_m, g_m$  be continuous functions  $f_i, g_i : X \rightarrow \mathbb{R}$  with  $g_i(x) \neq 0$  for all  $x \in X, i = 1, \dots, m$  and  $\int g_i d\mu \neq 0$ , for all  $\mu \in M(X, T), i = 1, \dots, m$ . If  $C \subseteq \mathbb{R}^m$  is closed and convex,  $\varphi \in C(X, \mathbb{R}^+)$ , then*

$$\begin{aligned}
& BS \left( \left\{ x \in X : A \left( \left( \frac{\sum_{k=0}^{n-1} f_j(T^k x)}{\sum_{k=0}^{n-1} g_j(T^k x)} \right)_{j \in \{1,2,\dots,m\}} \right) \subseteq C \right\}, \varphi \right) \\
& = \sup \left\{ \frac{h(T, \mu)}{\int \varphi d\mu} : \mu \in M(X, T), \left( \frac{\int f_i d\mu}{\int g_i d\mu} \right)_{i \in \{1,\dots,m\}} \in C \right\}.
\end{aligned}$$

### 5.3 symbolic space and iterated function systems

Consider a subshift of finite type  $\Sigma_A^+$  of the unilateral full shift on  $m$  symbols  $I = \{1, 2, \dots, m\}$  with  $m \geq 2$ . Let  $\sigma$  be the shift map, and  $A = (a_{ij})_{1 \leq i, j \leq m}$  be the transfer matrix of zeros and ones. In this section, we assume that  $A$  is an irreducible and aperiodic stochastic matrix, that is, there is some power  $m$  such that all the entries of  $A^m$  are strictly positive. This assumption implies the specification property.

For  $x = (x_i)_{i \geq 1}$  and  $y = (y_i)_{i \geq 1}$ , set  $\nu(x, y) = \inf\{i \geq 1 : x_i \neq y_i\}$ . Let  $\varphi$  be a strictly positive continuous function on  $\Sigma_A^+$ . Write  $S_n \varphi = \sum_{i=0}^{n-1} \varphi \circ \sigma^i$  for each  $n \geq 1$ . For  $x \neq y \in \Sigma_A^+$ , define

$$d_\varphi(x, y) = \begin{cases} 0, & x = y, \\ 1, & x_1 \neq y_1, \\ \exp(-\min_{\nu(x, z) \geq m} S_m \varphi(z)), & m = \nu(x, y). \end{cases}$$

Remark that given  $\Psi > 1$ , we can choose  $\varphi \equiv \ln \Psi$ ,  $d_\varphi$  is the metric in [15].

**Proposition 5.1.** *In  $(\Sigma_A^+, d_\varphi)$ , for any subset  $Z \subset \Sigma_A^+$ , we get  $\dim_H(Z) = BS(Z, \varphi)$ .*

Let  $\omega_{ij}$  be a Lipschitz contraction map on  $\mathbb{R}^n$  for each nonzero  $a_{ij}$ . There exists a unique vector  $E = (E_1, \dots, E_m)$  of non-empty compact subsets of  $\mathbb{R}^n$  satisfying  $E_i = \bigcup_{a_{ij}=1} \omega_{ij}(E_j)$ . The union  $E = \bigcup_{i=1}^m E_i$  is called a self-similar set for recurrent iterated function system  $\{\omega_{ij}, (a_{ij})\}$ .

Let  $F$  be a compact subset of  $E$ . Set  $F_i = F \cap E_i, i = 1, \dots, m$ , if vector  $(F_1, F_2, \dots, F_m)$  satisfying  $F_i \subseteq \bigcup_{a_{ij}=1} \omega_{ij}(E_j)$ , then the set  $F$  is called a sub-self-similar set for  $\{\omega_{ij}, (a_{ij})\}$ .

Assume that

- (i) Each map  $\omega_{ij}$  is a  $C^{1+\gamma}$  diffeomorphism.
- (ii)  $D\omega_{ij}$  is always a similarity map, i.e.,  $|(D\omega_{ij})_x(\nu)| = s_{ij}(x) \cdot |\nu|$  for each  $x, \nu \in \mathbb{R}^n$ .
- (iii)  $\{\omega_{ij}, (a_{ij})\}$  satisfies the open set condition [8].

Let  $\pi : \Sigma_A^+ \rightarrow E$  be given by

$$\pi(x) = \text{the only point in } \bigcap_{n \geq 1} \omega_{x_1 x_2} \omega_{x_2 x_3} \cdots \omega_{x_{n-1} x_n}(E_{x_n}).$$

The scale function of  $E$  is the map  $\psi : \Sigma_A^+ \rightarrow \mathbb{R}$  given by  $\psi(x) = \log s_{x_1 x_2}(\pi \sigma x)$ . Let  $\varphi(x) = -\psi(x)$ , then  $\varphi$  is a positive Hölder continuous function.

**Proposition 5.2.** *In  $(\Sigma_A^+, d_\varphi)$ , for any subset  $Z \subset \Sigma_A^+$ , we get  $\dim_H(\pi Z) = \dim_H(Z)$ .*

Combining Propositions 5.1 and 5.2, Corollaries 5.1, 5.9, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.10, 5.8 can hold about Hausdorff dimension in iterated function system with open set condition. We take Corollary 5.10 as an example.

**Corollary 5.11.** *Let  $f_1, g_1, \dots, f_m, g_m$  be continuous functions  $f_i, g_i : \Sigma_A^+ \rightarrow \mathbb{R}$  with  $g_i(x) \neq 0$  for all  $x \in \Sigma_A^+, i = 1, \dots, m$  and  $\int g_i d\mu \neq 0$ , for all  $\mu \in M(\Sigma_A^+, \sigma), i = 1, \dots, m$ . If  $C \subseteq \mathbb{R}^m$  is closed and convex, then*

$$\dim_H \left( \pi \left\{ x \in \Sigma_A^+ : A \left( \left( \frac{\sum_{k=0}^{n-1} f_j(\sigma^k x)}{\sum_{k=0}^{n-1} g_j(\sigma^k x)} \right)_{j \in \{1, 2, \dots, m\}} \right) \subseteq C \right\} \right) \\ = \sup \left\{ \frac{h(T, \mu)}{-\int \log s_{x_1 x_2}(\pi \sigma x) d\mu} : \mu \in M(\Sigma_A^+, \sigma), \left( \frac{\int f_i d\mu}{\int g_i d\mu} \right)_{i \in \{1, \dots, m\}} \in C \right\}.$$

Remark that our results are valid for sofic system (self-conformal function system) induced by a subshift of finite type modelled by a directed and strongly connected multi-graph. Similar to Corollary 5.11, we give a positive answer to the conjecture in [15] (see [19] for the view of Hausdorff dimension) from the view of topological pressure.

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